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NON-HYPERELLIPTIC RIEMANN SURFACES

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1. Let W be a Riemann surface of genus $g \geq 2$. Abel's Theorem gives an analytic embedding, with respect to an arbitrary base point, of W as a submanifold of its Jacobi variety $J(W)$ (see Gunning [1, p. 161]). Denote by W_r the linear equivalence classes in $J(W)$ of divisors consisting of r points in W. Then $W = W_1$ and W_r is a subvariety of dimension r for $0 \leq r \leq g$. More generally, let W_r^a denote the set of all linear equivalence classes of divisors of degree r in $J(W)$, which admit at least a linearly independent meromorphic function, or equivalently, the set of all line bundles on W of Chern class r which admit at least a linearly independent analytic section. Then $W_r = W_r^1$, and W_r^a is a subvariety of $J(W)$. Alan Mayer [4] showed that $\dim W_r^a \leq r - 2a + 2$ (provided $1 \leq r \leq g - 1$, $a \geq 2$, and $r - 2a + 2 \geq -1$) and that the maximum is in fact attained whenever W is hyperelliptic. He then conjectured that the converse was true, and this is our main result.

Theorem 1. *If W is not hyperelliptic, then* $\dim W_r^a \leq r - 2a + 1$ (*provided* $1 \leq r \leq g - 1, a \geq 2$, *and* $r - 2a + 1 \geq -1$).

Thus surfaces which are not hyperelliptic have fewer "special" divisors than those which are.

2. Before proceeding to the proof of Theorem 1, it may be of interest to see how two classical theorems on non-hyperelliptic surfaces may be deduced from this result.

Clifford's Theorem. *If W is not hyperelliptic, then no translate of W_r is contained in $-W_r$, for $1 \leq r \leq g - 2$.*

Proof. The set of all elements x such that the translate of $-W_r$ by x is contained in W_r is the set $W_r \ominus -W_r = W_{2r}^{r+1}$ (see § 3). Hence the theorem is equivalent to the assertion that if W is not hyperelliptic then W_{2r}^{r+1} is empty. If $2r \leq g - 1$, Theorem 1 states that $\dim W_{2r}^{r+1} \leq -1$. If $2r \geq g - 1$, let k be the divisor class in $J(W)$ of an abelian differential. Then $k - W_{2r}^{r+1} = W_{2g-2-2r}^{g-r}$ by the Riemann-Roch Theorem, and since $2 \leq 2g - 2 - 2r \leq g - 1$ this is the same as the first case.

Nöther's Theorem. *If W is not hyperelliptic, then every quadratic differential on W can be written as the sum of (three) products of abelian differentials.*

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Proof. The main step in the classical proof (see Hensel-Landsberg [3, p. 508]) is to find two abelian differentials ω_1 and ω_2 on W which vanish simultaneously (counting multiplicities) at precisely $g - 2$ points. By Theorem 1, $\dim W^2_{g-1} \leq g - 4$, so $\dim W^2_{g-1} \oplus -W_1 \leq g - 3 < \dim W_{g-2}$. Hence it is possible to pick $g - 2$ points $p_1, \cdots, p_{g-2}$ such that, for all $x \in W$, $p_1 + \cdots + p_{g-2} + x \notin W^2_{g-1}$, i.e. there is no non-constant meromorphic function on W with poles at $p_1, \cdots, p_{g-2}$ and x (counting multiplicities). Let ω_1 and ω_2 be two linearly independent abelian differentials vanishing at $p_1, \cdots, p_{g-2}$. Then by the Riemann-Roch Theorem ω_1 and ω_2 can have no other common zeroes, and moveover every abelian differential which vanishes at $p_1, \cdots, p_{g-2}$ must be a linear combination of ω_1 and ω_2.

Choose an abelian differential ω_3 with no zeroes in common with either ω_1 or ω_2. Suppose that α_1, α_2. and α_3 are three abelian differentials such that $\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = 0$. It follows from the above remarks that α_3 is a linear combination of ω_1 and ω_2, since it vanishes at $p_1, \cdots, p_{g-2}$. Suppose then that $\alpha_1\omega_1 + \alpha_2\omega_2 = 0$. Then α_2 vanishes at the g points other than $p_1, \cdots, p_{g-2}$ where ω_1 vanishes. But by the Riemann-Roch Theorem any abelian differential vanishing at these points must be a multiple of ω_1. Therefore there are $3g - 3$ linearly independent quadratic differentials of the form $\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3$. But by the Riemann-Roch Theorem there are only $3g - 3$ linearly independent quadratic differentials in all; hence every quadratic differential can be written in the form $\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3$.

3. In this section we recall some of the basic properties of the subvarieties W^a_r, and calculate their dimension in the case where W is hyperelliptic. All the results in this section occur in Mayer [4].

If A and B are subvarieties of a complex torus J, we define new subvarieties:

$$-A = \{-a \mid a \in A\} ,$$
$$A \oplus B = \{a + b \mid a \in A, b \in B\} ,$$
$$A \ominus B = \{c \mid c + B \subseteq A\} = \cap\{A - b \mid b \in B\} .$$

Here $-A$ is a subvariety since it is the inverse image of A under multiplication by -1; $A \oplus B$ is a subvariety because it is the image of $A \times B$ under addition $\oplus$: $J \times J \to J$, which is a proper map (see the "Proper Mapping Theorem", Gunning and Rossi [2, p. 162]); finally, $A \ominus B$ is the intersection of the subvarieties $A - b$, and hence is itself a subvariety.

Theorem 2. $\dim W_r = r$ *(for* $0 \leq r \leq g$*).*

Proof. $W_0 = \{0\}$, $W_1 = W$, and $W_g = J(W)$ by Abel's Theorem, so Theorem 1 is true for $r = 0, 1, g$. In general $W_{r+1} = W_r \oplus W_1$, so $\dim W_{r+1} \leq \dim W_r + 1$. But since the extreme cases are known, we must have $\dim W_r = r$ for $0 \leq r \leq g$.

Lemma 1. $W^a_r \ominus -W_1 = W^{a+1}_{r+1}$ *(for* $r \geq 0, a \geq 1$*).*

Proof. If a divisor class admits at least $a + 1$ linearly independent meromorphic functions, there will be at least a linearly independent combination of them which vanish at each point. Hence $W^{a+1}_{r+1} \oplus -W_1 \subseteq W^a_r$, which implies $W^{a+1}_{r+1} \subseteq W^a_r \ominus -W_1$. On the other hand, if $d \in W^a_r \ominus -W_1$, then for every point $x \in W$ the divisor class $d-x$ admits at least a linearly independent meromorphic function. Hence among the meromorphic functions admitted by the divisor d, we can find, for each $x \in W$ which is not a point of d, at least a linearly independent one which vanishes at x. If the divisor d did not admit at least $a+1$ linearly independent meromorphic functions, this would be impossible since then all the meromorphic functions admitted by d would vanish everywhere, but $a \geq 1$. Therefore $W^a_r \ominus -W_1 \subseteq W^{a+1}_{r+1}$.

Let k be the divisor class of an abelian differential.

Theorem 3.

i) $k - W^a_r = W^b_s$, *where* $s = 2g-2-r$ *and* $b = g-1-r+a$.

ii) $W^a_{r-s} \ominus -W_s = W^{a+s}_r$ *for* $0 \leq s \leq r, a \geq 1$.

iii) $W^a_r \ominus W_s = W^a_{r-s}$ *for* $0 \leq s \leq r \leq g-1, a \geq 1$.

Proof. (i) is the Riemann-Roch Theorem. (ii) is true for $s = 0$ (trivially) and for $s = 1$ (by Lemma 1). Suppose it is true for $s = t \geq 1$. Then

$$\begin{aligned} W^a_{r-t-1} \ominus -W_{t+1} &= W^a_{r-t-1} \ominus [-W_1 \oplus -W_t] \\ &= [W^a_{r-t-1} \ominus -W_1] \ominus -W_t = W^{a+1}_{r-t} \ominus -W_t = W^{a+t+1}_r \,. \end{aligned}$$

Hence it is also true for $s = t+1$. Therefore by induction it is true for all s. To prove (iii), let $t = 2g-2-r$ and $b = g-1-r+a$. Then if $r \leq g-1$, b will be at least 1 and

$$k - [W^a_r \ominus W_s] = [k - W^a_r] \ominus -W_s = W^b_t \ominus -W_s = W^{b+s}_{t+s} = k - W^a_{r-s} \,.$$

Therefore $W^a_r \ominus W_s = W^a_{r-s}$.

Theorem 4. *Let* $1 \leq r \leq g-1$ *and* $a \geq 2$, *and suppose* W^a_r *is not empty. Then* $\dim W^a_{r-1} + 1 \leq \dim W^a_r \leq \dim W^{a-1}_{r-1} - 1$.

Proof. Let V be an irreducible subvariety of W^a_{r-1} of maximal dimension; then $\dim V < g$. Since $W^a_{r-1} \oplus W_1 \subseteq W^a_r$, $V \oplus W_1 \subseteq W^a_r$. Moreover, $V \oplus W_1$ is irreducible, since it is the image of $V \times W_1$ under the addition map $\oplus$: $J(W) \times J(W) \to J(W)$. If $\dim V \oplus W_1$ were equal to $\dim V$, we would have $V = V \oplus W_1$, and by induction $V = V \oplus W_g$ which is impossible unless $V = \varnothing$ since $\dim V < g$. Therefore $\dim W^a_r \geq \dim W^a_{r-1} + 1$, if $W^a_r \neq \varnothing$.

But we also have $W^a_r \oplus -W_1 \subseteq W^{a-1}_{r-1}$. Let X be an irreducible subvariety of W^a_r of maximal dimension; then $0 \leq \dim X < g$. Since $X \oplus -W_1 \subseteq W^{a-1}_{r-1}$ we conclude as before that $\dim X \oplus -W_1 > \dim X$ and hence $\dim W^a_r \leq \dim W^{a-1}_{r-1} - 1$.

Theorem 5. *Let* $a \geq 1$, $1 \leq r \leq g-1$, *and* $r - 2a + 2 \geq -1$. *Then* $\dim W^a_r \leq r - 2a + 2$.

Proof. Applying the right hand side of the previous Theorem inductively, we deduce that

$$\dim W_r^a \leq \dim W_{r-a+1} - (a-1) = r - 2a + 2 ,$$

provided that W_r^a is not empty.

Theorem 6. *If W is hyperelliptic, and r and a are as above, then* $\dim W_r^a = r - 2a + 2$.

Proof. Let d be the divisor class of a meromorphic function f on W of order 2. Then the divisor class nd admits $n+1$ linearly independent meromorphic functions $1, f, f^2, \cdots, f^n$. Therefore $\dim W_{2n}^{n+1} \geq 0$. But applying the left hand side of Theorem 4 inductively we deduce that

$$\dim W_r^a \geq \dim W_{2a-2}^a + r - 2a + 2 \geq r - 2a + 2 ,$$

provided that $r - 2a + 2 \geq 0$.

4. We now observe that it is sufficient to prove Theorem 1 in the case $a = 2$. For if W_r^a is empty there is nothing to prove, and W_r^a will always be empty if $a \geq 2$ and $r = 1$. Otherwise we may apply the right hand side of Theorem 4 inductively to prove that $\dim W_r^a \leq \dim W_{r-a+2}^2 - (a-2)$. Now if Theorem 1 is true for $a = 2$, $\dim W_{r-a+2}^2 \leq r - a - 1$, which proves $\dim W_r^a \leq r - 2a + 1$. Thus in general deficiencies in the dimensions of the W_r^2 will propagate themselves upward.

We may now reinterpret Theorem 1 by means of the following observation.

Theorem 6. $\dim W_r^2 = r - 2$ *if and only if* $W_{r-1} = W_r^2 \oplus -W_1$ *(for* $2 \leq r \leq g-1$).

Proof. Since $W_r^2 = W_{r-1} \ominus -W_1$, we always have $W_r^2 \ominus -W_1 \subseteq W_{r-1}$. But W_{r-1} is an irreducible subvariety, being the image under addition of a product of irreducible subvarieties (i.e. a product of $r-1$ copies of W). Therefore unless $W_r^2 \oplus -W_1$ is equal to W_{r-1}, its dimension must be strictly smaller. On the other hand, the proof of Theorem 4 shows that $\dim W_r^2 < \dim W_r^2 \oplus -W_1$. Therefore $\dim W_r^2 \leq r-3$ unless $W_r^2 \oplus -W_1 = W_{r=1}$.

Hence, to complete the proof of Theorem 1, it is sufficient to prove the following theorem.

Theorem 7. *If $W_r = W_{r+1}^2 \oplus -W_1$ for some integer r, $1 \leq r \leq g-2$, then W is hyperelliptic.*

5. To prove Theorem 7, let r be the smallest integer with $1 \leq r \leq g-2$ such that the hypothesis holds. If $r = 1$, then W_2^2 is not empty and W is hyperelliptic. We will show that if $r > 1$ we get a contradiction.

Let $W_{(t)}$ be the Cartesian product of W_1 with itself t times. An index of order r is defined to be an unordered collection $A = \{i_1, \cdots, i_r\}$ of r distinct elements of the set $\{1, \cdots, g-1\}$. Write $|A| = r$ and $p_A = p_{i_1} + \cdots + p_{i_r}$ for points $p_1, \cdots, p_{g-1}$.

Let S be the subset of $W_{(g-1)}$ of all points $(p_1, \cdots, p_{g-1})$ such that either

1) $p_1 + \cdots + p_{g-1} \in W_{g-1}^2$,

2) $p_i = p_j$ for some $i \neq j$,
3) $p_A \in W_r^2$ for some A with $|A| = r$,
4) $p_B \in W_r^2 \oplus -W_1$ for some B with $|B| = r - 1$, or
5) $p_C + p_j \in W_{r+1}^2$ for some C with $|C| = r$.

Lemma 2. *The set S is a proper subvariety of $W_{(g-1)}$.*

Proof. Conditions (1) and (3) define proper subvarieties since $\dim W_r^2 < \dim W_r$; condition (4) does since $W_{r-1} \not\subseteq W_r^2 \oplus -W_1$ by the minimal choice of r. For condition (5), if $j \notin C$, this is the same as condition (3) for $A = C \cup j$; while if $j \in C$, write $C = j \cup D$. Suppose $2p_j + p_D \in W_{r+1}^2$ for all $p \in W_{(g-1)}$. Then $2W_1 \oplus W_{r-1} \subseteq W_{r+1}^2$, or $2W_1 \subseteq W_{r+1}^2 \ominus W_{r-1} = W_2^2$, so that W_2^2 is not empty and W is hyperelliptic. Otherwise condition (5) must determine a proper subvariety. Since $W_{(g-1)}$ is irreducible, the union of a finite number of proper subvarieties will again be a proper subvariety.

Next let Z be the subvariety of $W_{(g-1)} \times W_1$ of all points $(p_1, \cdots, p_{g-1}, q)$ such that for some indices $A \neq B$, both of length r, $p_A + q \in W_{r+1}^2$ and $p_B + q \in W_{r+1}^2$. Let Q be the projection of Z onto $W_{(g-1)}$. Q is a subvariety by the "Proper Mapping Theorem" quoted above.

Lemma 3. *Q is a proper subvariety of $W_{(g-1)}$.*

Proof. Let $A \neq B$ be two indices of length r. Let $k \in A$ but $k \notin B$. Write $A = k \cup L$ with $|L| = r - 1$, $k \notin L$. Choose $p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_{g-1}$ so that $p_L \notin W_r^2 \oplus -W_1$ and $p_B \notin W_r^2$. There will be only finitely many q with $p_B + q \in W_{r+1}^2$ (for otherwise $p_B \oplus W_1 \subseteq W_{r+1}^2$ which implies $p_B \in W_{r+1}^2 \ominus W_1 = W_r^2$). Then $p_L + q \notin W_r^2$ so by the same argument there are only finitely many choices of q and s with $p_L + q + s \in W_{r+1}^2$. Choose p_k to be not one of these s. Then if $p_B + q \in W_{r+1}^2$, $p_A + q \notin W_{r+1}^2$. Thus the image of the subvariety $Z_{AB} = \{(p, q) \in W_{(g-1)} \times W_1 | p_A + q \in W_{r-1}^2$ and $p_B + q \in W_{r-1}^2\}$ in $W_{(g-1)}$ does not contain p. Since Q is the union of the images of the Z_{AB}, Q is proper.

Lemma 4. *If $p \in W_r$, there exists a $q \in W_1$ with $p + q \in W_{r+1}^2$.*

Proof. $W_r \subseteq W_{r+1}^2 \oplus -W_1$. Thus we can write $p = r - q$ with $r \in W_{r+1}^2$ and $q \in W_1$, and $p + q = r$.

Now choose $p = (p_1, \cdots, p_{g-1}) \in W_{(g-1)}$ with $p \notin S$ and $p \notin Q$. For each A with $|A| = r$ choose by Lemma 5 a point $q_A \in W_1$ with $p_A + q_A \in W_{r+1}^2$. Since $p_A + p_j \notin W_{r+1}^2$ for all j, $q_A \neq p_j$ for all j. Also if $|B| = r$ but $A \neq B$, then $q_A \neq q_B$, for otherwise we would have $(p, q_A) \in Z$ which cannot happen since $p \notin Q$. Since $p_1 + \cdots + p_{g-1} \notin W_{g-1}^2$, there exists a non-zero abelian differential ω which vanishes at $p_1, \cdots, p_{g-1}$, and this condition determines ω uniquely up to a constant factor. But since $p_A + q_A \in W_{r+1}^2$, there exist $g - r$ linearly independent abelian differentials vanishing at $p_{i_1}, \cdots, p_{i_r}, q_A$ for each $A = \{i_1, \cdots, i_r\}$. Hence there is a non-zero abelian differential vanishing at $p_1, \cdots, p_{g-1}$ and at q_A, which is therefore a non-zero multiple of ω. Hence ω vanishes at q_A for each index A of length r. Since $p_1, \cdots, p_{g-1}$ and the q_A are all distinct, and there are $\binom{g-1}{r}$ indices of length r, we get a contradiction if $\binom{g-1}{r} > g-1$. Hence either $r = 1$ or $r = g-2$. If $r = 1$ the surface is hyperelliptic as we saw before. Suppose $r = g-2$. If $g = 3$ we are done also. Suppose $g \geq 4$.

Lemma 5. *Let X and Y be analytic spaces and V a proper subvariety of $X \times Y$. Then there exists a proper subvariety S of X such that if $x \notin S$ then $T(x) = \{y \in Y \mid (x, y) \in V\}$ is a proper subvariety of Y.*

Proof. For each $y \in Y$, let $S(y) = \{x \in X \mid (x, y) \in V\}$. Then $S(y)$ is a subvariety of X, as is $S = \bigcap_{y \in Y} S(y)$. Since V is proper, some pair $(x, y) \notin V$, and for this pair $x \notin S(y)$, so $x \notin S$. Hence S is a proper subvariety of X. If $x \notin S$, then $x \notin S(y)$ for some y, and this pair $(x, y) \notin V$, so $y \notin T(x) = \{y \in Y \mid (x, Y) \in V\}$ and $T(x)$ is a proper subvariety of Y.

Now regard $W_{(g-1)} = W_{(g-2)} \times W_1$, and let L be a subvariety of $W_{(g-2)}$ such that if $(p_1, \cdots, p_{g-2}) \notin L$ then there exists a point p_{g-1} with $(p_1, \cdots, p_{g-2}, p_{g-1}) \notin S \cup Z$. This is possible by the previous lemma. Then the abelian differential ω constructed previously will vanish at the $2g-2$ distinct points $p_1, \cdots, p_{g-1}, q_1, \cdots, q_{g-1}$, where we let $q_i = q_{A_i}$ with $A_i = \{1, \cdots, i-1, i+1, \cdots, g-1\}$ as the index obtained by omitting i. We chose q_{g-1} so that $p_1 + \cdots + p_{g-2} + q_{g-1} \in W^2_{g-1}$, and we see now that if $p_1 + \cdots + p_{g-2} + y \in W^2_{g-1}$, then there exist two linearly independent abelian differentials vanishing at $p_1, \cdots, p_{g-2}$ and y, and thus there is a non-zero abelian differential vanishing at $p_1, \cdots, p_{g-2}$, y and p_{g-1}. But this must be a non-zero multiple of ω. Thus ω will vanish at y. Since ω has precisely $2g-2$ distinct zeroes, y must be one of the points $p_1, \cdots, p_{g-1}, q_1, \cdots, q_{g-1}$. If $y = p_j$ we have $p_{A_{g-1}} + p_j \in W^2_{g-1}$ which contradicts condition (5) of the definition of S. If $y = q_j$ with $j \neq g-1$, then $p_{A_{g-1}} + y \in W^2_{g-1}$ and $p_{A_j} + y \in W^2_{g-1}$ which contradicts the assumption that $(p_1, \cdots, p_{g-1}) \notin Z$. Hence q_{g-1} is the unique point in W_1 with $p_1 + \cdots + p_{g-2} + q_{g-1} \in W^2_{g-1}$; this proves

Lemma 6. *There exists a proper subvariety L of $W_{(g-2)}$ such that if $(p_1, \cdots, p_{g-2}) \notin L$ there is a unique point q_{g-1} with $p_1 + \cdots + p_{g-2} + q_{g-1} \in W^2_{g-1}$.*

Now fix a choice of $(p_1, \cdots, p_{g-3})$ with $p_1 + \cdots + p_{g-3} \notin W^2_{g-2} \oplus -W_1$ so that for at least one point p_{g-2} we have $(p_1, \cdots, p_{g-3}, p_{g-2}) \notin L$. This is possible by Lemma 6. Let $R = \{(p_{g-2}, p_{g-1}) \in W_1 \times W_1 \mid p_1 + \cdots + p_{g-1} \in W^2_{g-1}\}$. Then there are only finitely many p_{g-2} for which $(p_1, \cdots, p_{g-2}) \in L$, and for all other choices of p_{g-2} there is a unique p_{g-1} with $(p_{g-2}, p_{g-1}) \in R$. Hence R is a subvariety of dimension 1, for it is proper and cannot be a finite set since it projects onto an infinite set in the first factor. Moreover, there, must be an irreducible component N which projects onto the first factor, and this must be unique. In fact, N is a 1-sheeted branched cover of W_1 under projection on the first factor. Consequently the projection of N onto the first factor is one-to-one.

Lemma 7. *N also projects onto the second factor.*

Proof. If not, then N projects onto a single point p_{g-1}. This means that $p_1 + \cdots + p_{g-3} + p_{g-2} + p_{g-1} \in W^2_{g-1}$ for every p_{g-2}. Then $p_1 + \cdots + p_{g-3} + p_{g-1} \in W^2_{g-1} \ominus W_1 = W^2_{g-2}$, and $p_1 + \cdots + p_{g-3} \in W^2_{g-2} \oplus -W_1$, contrary to their choice.

Since the definition of R is symmetric in p_{g-2} and p_{g-1}, the projection of N onto the second factor must also be one-to-one. Thus N is the graph of a bi-analytic automorphism of W onto itself. Let G be the group of all such maps. Since $g \geq 4$, the group G is finite. Thus we have established

Lemma 8. *There exists a proper subvariety M of $W_{(g-3)}$ such that if $(p_1, \cdots, p_{g-3}) \notin M$, $p_1 + \cdots + p_{g-3} + p_{g-2} + p_{g-1} \in W^2_{g-1}$, and $(p_1, \cdots, p_{g-2}) \notin L$, then $p_{g-1} \in Gp_{g-2}$ where $Gp_{g-2} = \{g(p_{g-2}) \mid g \in G\}$.*

Now choose $(p_1, \cdots, p_{g-2})$ so that

1) $(p_1, p_2, \cdots, p_{g-2}) \notin L$,

2) $(p_1, p_2, \cdots, p_{g-4}, p_{g-2}, p_{g-3}) \notin L$,

3) $(p_1, p_2, \cdots, p_{g-3}) \notin M$, $(p_1, \cdots, p_{g-4}, p_{g-2}) \notin M$,

4) $p_{g-3} \notin Gp_{g-2}$.

Then there exists a p_{g-1} with $p_1 + \cdots + p_{g-1} \in W^2_{g-1}$ since $W_{g-2} \subset W^2_{g-1} \oplus -W_1$. Also by Lemma 8, $p_{g-1} \in Gp_{g-2}$ and $p_{g-1} \in Gp_{g-3}$. Thus $p_{g-3} \in Gp_{g-2}$ which is a contradiction. This proves $r \neq g-2$ and completes the proof of Theorem 7.

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